Farey Sequences

Definition

A Farey sequence $F(n)$ is a complete sequence of reduced form rational fractions in the range $[0..1]$ of the form $F(n) = \left\{ \frac{0}{1} < \frac{a_0}{b_0} < .....\frac{a_p}{b_p} < \frac{1}{1} \right\}$ with $b_j \le n$ for $\forall j$. The sequence is complete in the sense that, given a Farey sequence as defined above, we can find no rational fraction $\frac{p}{q}$ with $q \le n$ such that $\frac{a_j}{b_j} < \frac{p}{q} < \frac{a_{j+1}}{b_{j+1}}$.

Theorem

Given a Farey sequence $F(n)$, the following equality holds for $0 \le j \le p$:

$$\frac{a_{j-1} + a_{j+1}}{b_{j-1} + b_{j+1}} = \frac{a_j}{b_j}$$

where $a_{-1} = 0$, $a_{p+1} = 1$, $b_{-1} = 1$, $b_{p+1} = 1$.

Proof

Proof is by induction. For $n = 2$, we have the trivial Farey sequence $F(2) = \left\{ \frac{0}{1}, \frac{1}{2}, \frac{1}{1} \right\}$ where the above equality holds for $j = 0$. Given a Farey sequence $F(n)$, the sequence $F(n+1)$ comprises a merge of $F(n)$ and the sequence $F'(n+1) = \left\{ \frac{k_0}{n+1}, \frac{k_1}{n+1} ..... \frac{k_{p_{n+1}}}{n+1} \right\}, 0 < k_i < n+1 : \forall i$. If some $\frac{k_i}{n+1} \in F'(n+1)$ is not in reduced form then it can be represented as a reduced form rational fraction $\frac{k'}{n'}$ with $k' < n' \le n$. But this fraction will already be a member of $F(n)$ since $F(n)$ is complete. Thus only reduced form fractions will be members of $F'(n+1)$ and will be merged with $F(n)$. The merge is such that all elements of $F'(n+1)$ are inserted in their correct position in the sequence $F(n)$ in order to produce $F(n+1)$. (This observation enables an algorithm for the construction of $F(n)$ for any n to be designed.)

The complete proof will use the following 4 lemmas.

Lemma 1

There exists integers a, b, k and n such that $\frac{k}{n} < \frac{a}{b} < \frac{k+1}{n}$ for $n > 2,\ 0 < k < n-1,\ a < b$

Proof

Choose $a = k, b = n-1$. It is easy to verify that $\frac{k}{n} < \frac{k}{n-1} < \frac{k+1}{n}$.

Lemma 2

After merging, no two consecutive members of $F'(n+1)$ will be consecutive members of $F(n+1)$. Thus we can always find some fraction $\frac{a_j}{b_j}$ such that $\frac{k_i}{n+1} < \frac{a_j}{b_j} < \frac{k_{i+1}}{n+1}$ and where $a_j < b_j \le n$.

Proof

This follows directly from Lemma 1 since $k_{i+1} \ge k_i + 1$.

Lemma 3

Given a Farey sequence $F(n) = \left\{ \frac{0}{1} < \frac{a_0}{b_0} < .....\frac{a_p}{b_p} < \frac{1}{1} \right\}$, for any $-1 \le j \le p$, $a_{j+1}b_j - b_{j+1}a_j = 1$.

Proof

This follows directly from the theorem $\frac{a_{j-1} + a_{j+1}}{b_{j-1} + b_{j+1}} = \frac{a_j}{b_j}$.

$$\frac{a_{j-1} + a_{j+1}}{b_{j-1} + b_{j+1}} = \frac{a_j}{b_j} \Rightarrow a_j b_{j-1} + a_j b_{j+1} = a_{j-1}b_j + a_{j+1}b_j \Rightarrow a_j b_{j-1} - a_{j-1}b_j = a_{j+1}b_j - a_j b_{j+1}$$

This applies for all $0 \le j \le p$ and can be applied successively to create a sequence $\left\{ a_0 b_{-1} - a_{-1}b_0 = a_1 b_0 - a_0 b_1 = ...a_{j+1}b_j - a_j b_{j+1} = ...a_{p+1}b_p - a_p b_{p+1} \right\}$.

But $a_{-1} = 0, a_0 = 1, b_{-1} = 1, b_0 = n \Rightarrow a_0 b_{-1} - a_{-1}b_0 = 1$ and hence $a_{j+1}b_j - a_j b_{j+1} = 1$ for $-1 \le j \le p$.

From Lemma 2 we have some $\frac{k_i}{n+1} \in F'(n+1)$ such that $\frac{a_j}{b_j} < \frac{k_i}{n+1} < \frac{a_{j+1}}{b_{j+1}}$ are consecutive members of $F(n+1)$ and where $\frac{a_j}{b_j}$ and $\frac{a_{j+1}}{b_{j+1}}$ are consecutive members of $F(n)$ and $\frac{k_i}{n+1}$ is in reduced form.

Consider the rational fractions $\frac{k_i - a_{j+1}}{n+1-b_{j+1}}, \frac{k_i - a_j}{n+1-b_j}$. We can prove the following lemma.

Lemma 4

$$\frac{k_i - a_{j+1}}{n+1-b_{j+1}} = \frac{a_j}{b_j},\ \frac{k_i - a_j}{n+1-b_j} = \frac{a_{j+1}}{b_{j+1}}$$

Proof

$$\frac{a_{j+1}}{b_{j+1}} > \frac{k_i}{n+1} \Rightarrow k_i b_{j+1} < a_{j+1}(n+1) \Rightarrow k_i(n+1) - a_{j+1}(n+1) < k_i(n+1) - k_i b_{j+1} \Rightarrow \frac{k_i}{n+1} > \frac{k_i - a_{j+1}}{n+1-b_{j+1}}$$

Assume $\frac{k_i - a_{j+1}}{n+1-b_{j+1}} < \frac{a_j}{b_j}$

$$\frac{k_i - a_{j+1}}{n+1-b_{j+1}} < \frac{a_j}{b_j} \Rightarrow k_i b_j - a_{j+1} b_j < a_j(n+1) - a_j b_{j+1}$$
$$\Rightarrow k_i b_j - a_j(n+1) < a_{j+1} b_j - a_j b_{j+1} \Rightarrow k_i b_j - a_j(n+1) < 1$$

which follows from Lemma 3. But $\frac{k_i}{n+1} > \frac{a_j}{b_j} \Rightarrow k_i b_j - a_j(n+1) > 0$

Since $k_i b_j - a_j(n+1)$ is an integer, our original assumption is false and $\frac{k_i - a_{j+1}}{n+1-b_{j+1}} \geq \frac{a_j}{b_j}$. Since $\frac{k_i - a_{j+1}}{n+1-b_{j+1}} \in F(n)$, $\frac{k_i - a_{j+1}}{n+1-b_{j+1}} = \frac{a_j}{b_j}$ because there is no rational fraction $\frac{p}{q} \in F(n)$ such that $\frac{a_j}{b_j} < \frac{p}{q} < \frac{k_i}{n+1}$ as $F(n)$ is complete.

The second equality can be proved using a similar technique

$$\frac{a_j}{b_j} < \frac{k_i}{n+1} \Rightarrow k_i b_j > a_j(n+1) \Rightarrow k_i(n+1) - k_i b_j < k_i(n+1) - a_j(n+1) \Rightarrow \frac{k_i}{n+1} < \frac{k_i - a_j}{n+1-b_j}$$

Assume $\frac{k_i - a_j}{n+1-b_j} > \frac{a_{j+1}}{b_{j+1}}$

$$\frac{k_i - a_j}{n+1-b_j} > \frac{a_{j+1}}{b_{j+1}} \Rightarrow k_i b_{j+1} - a_j b_{j+1} > a_{j+1}(n+1) - a_{j+1} b_j$$

$$\Rightarrow a_{j+1} b_j - a_j b_{j+1} > a_{j+1}(n+1) - k_i b_{j+1} \Rightarrow a_{j+1}(n+1) - k_i b_{j+1} < 1$$

which follows from Lemma 3. But $\frac{k_i}{n+1} < \frac{a_{j+1}}{b_{j+1}} \Rightarrow a_{j+1}(n+1) - k_i b_{j+1} > 0$

Since $a_{j+1}(n+1) - k_i b_{j+1}$ is an integer, our original assumption is false and $\frac{k_i - a_j}{n+1-b_j} \le \frac{a_{j+1}}{b_{j+1}}$. Since $\frac{k_i - a_j}{n+1-b_j} \in F(n)$, $\frac{k_i - a_j}{n+1-b_j} = \frac{a_{j+1}}{b_{j+1}}$ because there is no rational fraction $\frac{p}{q} \in F(n)$ such that $\frac{k_i}{n+1} < \frac{p}{q} < \frac{a_{j+1}}{b_{j+1}}$ as $F(n)$ is complete.

This proves Lemma 4.

From Lemmas 3 and 4, the proof of the theorem follows directly.

$$\frac{k_i - a_j}{n+1-b_j} = \frac{a_{j+1}}{b_{j+1}} \Rightarrow k_i b_{j+1} - a_j b_{j+1} = a_{j+1}(n+1) - b_j a_{j+1} \Rightarrow a_{j+1}(n+1) - k_i b_{j+1} = 1$$

$$\frac{k_i - a_{j+1}}{n+1-b_{j+1}} = \frac{a_j}{b_j} \Rightarrow k_i b_j - a_{j+1} b_j = a_j(n+1) - a_j b_{j+1} \Rightarrow k_i b_j - a_j(n+1) = 1$$

$$k_i b_j - a_j(n+1) = a_{j+1}(n+1) - k_i b_{j+1} \Rightarrow \frac{k_i}{n+1} = \frac{a_j + a_{j+1}}{b_j + b_{j+1}}$$

This shows that the rational fraction $\frac{k_i}{n+1}$ inserted into $F(n)$ obeys the equality given in the Theorem. However, we still have to consider preceding values to check whether the Theorem is generally applicable to $F(n+1)$.

There are 2 possibilities: $\frac{a_{j-1}}{b_{j-1}} < \frac{a_j}{b_j} < \frac{k_i}{n+1} < \frac{a_{j+1}}{b_{j+1}}$ and $\frac{a_{j-1}}{b_{j-1}} < \frac{k_{i-1}}{n+1} < \frac{a_j}{b_j} < \frac{k_i}{n+1} < \frac{a_{j+1}}{b_{j+1}}$

where $\frac{a_{j-1}}{b_{j-1}}, \frac{a_j}{b_j}, \frac{a_{j+1}}{b_{j+1}}$ are consecutive terms in sequence $F(n)$.

Taking the first possibility, $\frac{a_{j-1}}{b_{j-1}} < \frac{a_j}{b_j} < \frac{k_i}{n+1} < \frac{a_{j+1}}{b_{j+1}}$

$\frac{a_{j-1}+a_{j+1}}{b_{j-1}+b_{j+1}} = \frac{a_j}{b_j} \Rightarrow a_{j-1}+a_{j+1} = pa_j,\ b_{j-1}+b_{j+1} = pb_j$ for some integer p.

$\frac{a_j+a_{j+1}}{b_j+b_{j+1}} = \frac{k_i}{n+1} \Rightarrow a_j+a_{j+1} = k_i,\ b_j+b_{j+1} = (n+1)$ because $b_j+b_{j+1} < 2(n+1)$

$\Rightarrow a_{j-1}-a_j = pa_j - k_i,\ b_{j-1}-b_j = pb_j-(n+1)$
$\Rightarrow a_{j-1}+k_i = a_j(1+p),\ b_{j-1}+(n+1) = b_j(1+p)$
$\Rightarrow \frac{a_{j-1}+k_i}{b_{j-1}+(n+1)} = \frac{a_j}{b_j}$

Thus the equality holds for when $\frac{k_i}{n+1}$ is the third term in the triplet. An identical proof shows that the equality holds when $\frac{k_i}{n+1}$ is the first term of the triplet and so $\frac{k_i+a_{j+2}}{(n+1)+b_{j+2}} = \frac{a_{j+1}}{b_{j+1}}$

Taking the second possibility $\frac{a_{j-1}}{b_{j-1}} < \frac{k_{i-1}}{n+1} < \frac{a_j}{b_j} < \frac{k_i}{n+1} < \frac{a_{j+1}}{b_{j+1}}$,

$\frac{a_{j-1}+a_{j+1}}{b_{j-1}+b_{j+1}} = \frac{a_j}{b_j} \Rightarrow a_{j-1}+a_{j+1} = pa_j,\ b_{j-1}+b_{j+1} = pb_j$ for some integer p.

$\frac{a_{j-1}+a_j}{b_{j-1}+b_j} = \frac{k_{i-1}}{n+1} \Rightarrow a_{j-1}+a_j = k_{i-1},\ b_{j-1}+b_j = (n+1)$

$\frac{a_j+a_{j+1}}{b_j+b_{j+1}} = \frac{k_i}{n+1} \Rightarrow a_j+a_{j+1} = k_i,\ b_j+b_{j+1} = (n+1)$

$k_i+k_{i-1} = a_{j-1}+2a_j+a_{j+1} = k_i,\ n+1 = \frac{b_{j-1}+2b_j+b_{j+1}}{2}$
$\Rightarrow k_i+k_{i-1} = a_j(2+p),\ n+1 = \frac{b_j(2+p)}{2}$
$\Rightarrow \frac{k_i+k_{i-1}}{2(n+1)} = \frac{a_j}{b_j}$

Thus the equality holds in this case also.

We have thus proved by induction that if the equality holds for $F(n)$, then it must hold for $F(n+1)$ which completes the proof of the Theorem.